Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 21

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Textbook Sections: 11.6

Todays Topic: The Ratio and Root Tests

First, a definition: Let $\sum_{n=1}^{\infty} a_n$ be a series, and let $\sum_{n=1}^{\infty} |a_n|$ be the series "of its absolute values."

1. $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.
2. $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if $\begin{cases} \sum_{n=1}^{\infty} a_n \text{ converges, and} \\ \sum_{n=1}^{\infty} |a_n| \text{ diverges.} \end{cases}$

Fact: If $\sum_{n=1}^{\infty} |a_n|$ converges (so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent), then $\sum_{n=1}^{\infty} a_n$ is itself convergent.

Pf: $-|a_n| \leq a_n \leq |a_n| \rightarrow -\sum |a_n| \leq \sum a_n \leq \sum |a_n|$.

Examples:

1. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is conditionally conv.
2. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ is absolutely conv.
3. $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is absolutely conv.

Today, we give some very useful tests for determining when a series is absolutely convergent.

Ratio Test:

Let $\sum_{n=1}^{\infty} a_n$ be a series, and compute

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \qquad \text{(assume } L \text{ exists!)}$$

1. If $L<1$, then $\sum a_n$ is abs. conv.
2. If $L>1$ (including $L=\infty$), $\sum a_n$ is divergent.
3. If $L=1$, the test is inconclusive.

Root Test:

Let $\sum_{n=1}^{\infty} a_n$ be a series, and compute

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L \qquad \text{(assume here that } L \text{ exists!)}$$

1. If $L<1$, then $\sum a_n$ is abs. conv.
2. If $L>1$ (including $L=\infty$), $\sum a_n$ is divergent.
3. If $L=1$, the test is inconclusive.

General Pieces of Advice: Given $\sum_{n=1}^{\infty} a_n$,

1) First check $\lim_{n\to\infty} a_n$.
2) Is $\sum a_n$ in the form of a geometric or p-series?
3) Is $\sum a_n$ alternating? If so, apply the alternating series test
4) Is it easy to compare $a_n \le b_n$ or $a_n \ge c_n$ for a better understood series $\sum b_n$ or $\sum c_n$?
5) If one can write $a_n = f(n)$, try the integral test.
6) If a_n has factorials or products, try the ratio test
7) If a_n looks like $(b_n)^n$ for some b_n, try the root test

Examples:

(1) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$

Identify: $a_n = (-1)^n \frac{n^3}{3^n}$ is not clearly monotone (although it does become monotone eventually — so this does pass the alt. series test), but is built out of factorials and exponentials. Lets try the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \lim_{n\to\infty} \frac{1}{3}\left|\left(1+\frac{1}{n}\right)^3\right| = \frac{1}{3} < 1.$$

Thus, $\sum (-1)^n \frac{n^3}{3^n}$ converges!

(2) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Identify: $a_n = \frac{n^n}{n!}$.

Method 1: Try ratio test (b.c. of powers/exponentials)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1.$$

$\Rightarrow$ diverges!

Method 2: Use comparison $a_n = \frac{n\cdot\cdots\cdot n}{n\cdot(n-1)\cdot\cdots\cdot 1} \geq n \Rightarrow \lim_{n\to\infty} a_n = \infty$

(3) $\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n+6}\right)^n$

Identify: $a_n = (b_n)^n$ with $b_n = \frac{2n+5}{3n+6}$. Try root test

$$\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} b_n = \frac{2}{3} \Rightarrow \text{converges.}$$

(4) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$.

Identify: $a_n = (b_n)^n$ with $b_n = -\frac{2}{n}$. Try the root test:

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\left(-\frac{2}{n}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2}{n} = 0 \Rightarrow \text{converges.}$$

(5) ⚠ Warning ⚠ All p-series give inconclusive results: $a_n = \frac{1}{n^p}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^p = 1$$

(irrespective of whether $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges).

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{p/n}} = 1.$$

Proofs of the Tests: (Let's focus on the root test)

Idea: Use the limits to build convergent comparison series.

Case 1: Assume $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L < 1$.

Let's choose a number $r_0 \in (L, 1)$ to act as a "barrier" in the sense that $\exists N \geq 1$ s.t. $\forall n \geq N$,

$$|a_n|^{\frac{1}{n}} < r_0 < 1$$

(i.e., after time N, all of the $|a_n|^{\frac{1}{n}}$ are smaller than 1 by a definite amount).

Step 1.1 Bound: $|a_n|^{\frac{1}{n}} < r_0 \Rightarrow |a_n| < r_0^n$ for all $n \geq N$

Step 1.2 Compare: $\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} r_0^n \leq \sum_{n=0}^{\infty} r_0^n = \frac{1}{1-r_0}$

Thus, $\sum_{n=1}^{\infty} |a_n|$ converges.

Case 2: Assume $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L > 1$.

Similarly, pick $R_0 \in (1, L)$, and $N \geq 1$ s.t. $\forall n \geq N$, $|a_n|^{\frac{1}{n}} \geq R_0 > 1$.

Step 2.1 Bound: $|a_n|^{\frac{1}{n}} \geq R_0 \Rightarrow |a_n| \geq R_0^n$ for all $n \geq N$

Step 2.2 Compare: $\sum_{n=N}^{\infty} |a_n| \geq \sum_{n=N}^{\infty} R_0^n = \infty$

Thus, $\sum_{n=1}^{\infty} |a_n|$ diverges.

Challenge: Prove the ratio test using similar ideas.

You can check your work in Strat.